Solution to Assignment 7

Supplementary Problems

We assume $\varepsilon \in (0,1)$.

1. Study the uniform convergence for the following sequences of functions on the indicated intervals.

   (a) $\left\{\dfrac{x}{x+n}\right\}$; $\quad [0,\infty), \quad [0,12]$.

   (b) $\left\{\dfrac{x^n}{1+x^n}\right\}$; $\quad [0,\infty), \quad [0,1], \quad [2,5]$.

   Solution. (a) The pointwise limit is the zero function. Let $\{f_n\}$ be the sequence. We have $f_n'(x) = n/(x+n)^2 > 0$ which means the function is increasing and tends to 1 as $x \to \infty$. So $\|f_n - 0\|_\infty = \|f_n\|_\infty = 1$, which is not equal to zero. This sequence is not uniformly convergent to 0 on $[0,\infty)$. If now we restrict to $[0,12]$, the max of f_n is attained at $x = 12$, so now $\|f_n\|_\infty = 12/(12+n) \to 0$ as $n \to \infty$. We conclude that it is uniformly convergent on $[0,12]$.

   (b) The pointwise limit is the constant one for $x \in (1,\infty)$. We have

$$\frac{d}{dx}\left(1 - \frac{x^n}{1+x^n}\right) = \frac{d}{dx}\frac{1}{1+x^n} = \frac{-nx^{n-1}}{(1+x^n)^2} < 0, \quad x \in (0,\infty) ,$$

   so the supnorm is given by $\lim_{x\to 1} 1/(1+x^n) = 1/2$. It means $\|f_n - 1\|_\infty = 1/2 \neq 0$, so no uniform convergence on $(1,\infty)$. On the other hand, on $[2,5]$ the supnorm is attained at $x = 2$, so $\|f_n - 1\|_\infty = 1/(1+2^n) \rightrightarrows 0$.

2. Study the uniform convergence of the following sequence of functions on the indicated intervals by any method.

   (a) $\left\{\dfrac{nx}{1+n^2x^2}\right\}$; $\quad [0,\infty)$.

   (b) $\left\{\dfrac{\sin nx}{1+nx}\right\}$; $\quad [0,\infty), \quad [1,\infty)$.

   Solution. (a) The pointwise limit is the zero function. By taking derivative we see that the maximum of f_n is attained at $x = 1/n$. It follows that

$$\left\|\frac{nx}{1+n^2n^2} - 0\right\| = \frac{n \times 1/n}{1+n^2 \times 1/n^2} = \frac{1}{2} \neq 0 ,$$

   so the convergence is not uniform.

   (b) The pointwise limit is again the zero function. It is not good to determine the maximum of each function. But we observe that $f_n(\pi/(2n)) = 2/(2+\pi)$, so

$$\left\|\frac{\sin nx}{1+nx} - 0\right\| \geq f_n\left(\frac{\pi}{2n}\right) = \frac{2}{2+\pi} \neq 0 ,$$

   so the convergence is not uniform. In this case it is nice to draw an ε-tube with $\varepsilon = 1/4$, say, to visualize the situation.

3. Study the pointwise and uniform convergence of $\{n^\alpha x^\beta e^{-nx}\}$ on $[0,\infty)$ for $\alpha, \beta > 0$.

   Solution. (c) The pointwise limit is the zero function on $[0,\infty)$. We find the maximum of $f_n = n^\alpha x^\beta e^{-nx}$ by setting
$$0 = \frac{d}{dx} f_n(x) = n^\alpha \beta x^{\beta-1} e^{-nx} - n^{\alpha+1} x^\beta e^{-nx},$$
   which implies $x = \beta/n$. It is easy to check that this is the maximum as f_n is positive and tends to 0 at $x = 0$ and $x = \infty$. Therefore,
$$\|x^2 e^{-nx} - 0\| = f_n(\beta/n) = \beta^\beta n^{\alpha-\beta} e^{-\beta} \ ,$$
   which tends to 0 if and only if $\alpha < \beta$. We conclude that $\{n^\alpha x^\beta e^{-nx}\}$ uniformly converges to 0 on $[0,\infty)$ iff $\alpha < \beta$.

4. Find the pointwise limit of $\{(\cos \pi x)^{2n}\}$, $x \in (-\infty, \infty)$, and prove its uniform convergence on $[a,b]$ if $[a,b] \cap \mathbb{Z}$ is empty.

   Solution. The pointwise limit is $g(x) = 0$ for non-integer x and $g(x) = 1$ for $x \in \mathbb{Z}$. On any interval $[a,b]$ contained in $(n, n+1)$ for some integer n, by the continuity of the cosine function, $|\cos x|$ attains its maximum at some $c \in [a,b]$ so that $|\cos x| \le |\cos c| < 1$ on $[a,b]$. It follows that
$$\|(\cos x)^{2n} - 0\| \le c^n \to 0 \ ,$$
   independent of x. That is, the convergence is uniform on $[a,b]$.

5. Find the pointwise limit of $\{\text{Arctan } nx\}$ and show that this sequence of functions is uniformly convergent on $[a,\infty)$ for every positive a but not uniformly convergent on $(0,\infty)$. The function Arctan is the inverse function of the tangent function on $(-\infty,\infty)$ to $(-\pi/2, \pi/2)$.

   Solution. The function $\text{Arctan}(x)$ is strictly increasing and satisfies $\lim_{x\to\infty} \text{Arctan}(x) = \pi/2$. Its pointwise limit on $(0,\infty)$ is the constant function $\pi/2$. For $a > 0$, $\|\pi/2 - \text{Arctan } nx\| = \pi/2 - \text{Arctan } na \to 0$ as $n \to \infty$. So the convergence is uniform on $[a,\infty)$. However, the convergence is not uniform on $(0,\infty)$. For the supnorm of $\|f_n - \pi/2\|$ over $(0,\infty)$ is always equal to $\lim_{x\to 0} (\pi/2 - \text{Arctan } nx) = 0$.

6. Let $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. Prove that $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$ for $\alpha, \beta \in \mathbb{R}$.

   By assumption, for $\varepsilon > 0$, there is some n_1, n_2 such that
$$|f_n(x) - f(x)| < \frac{\varepsilon}{2 + 2|\alpha|} \ , \quad n \ge n_1, \ x \in E \ ,$$
   and
$$|g_n(x) - g(x)| < \frac{\varepsilon}{2 + 2|\beta|} \ , \quad n \ge n_2, x \in E \ .$$
   Therefore, for $n \ge n_0 \equiv \max\{n_1, n_2\}$,
$$\begin{aligned} |(\alpha f_n(x) - \beta g_n(x)) - (\alpha f(x) - \beta g(x))| &\le |\alpha||f_n(x) - f(x)| + |\beta||g_n(x) - g(x)| \\ &< |\alpha|\frac{\varepsilon}{2+2|\alpha|} + |\beta|\frac{\varepsilon}{2+2|\beta|} < \varepsilon \ . \end{aligned}$$

7. Let the two sequences of functions $\{f_n\}$ and $\{g_n\}$ uniformly converge to f and g respectively in E.

(a) Show that their product $\{f_n g_n\}$ converges to fg uniformly on E under the assumption that $\|f_n\| \leq M$, $\|g_n\| \leq N$ for all $n \geq 1$ for some M, N.

(b) Let $f_n(x) = x + 1/n \rightrightarrows f(x) = x$ on $(-\infty, \infty)$. Show that $\{f_n^2\}$ does not converge uniformly to f^2. It shows that the assumption in (a) cannot be dropped.

Solution. (a) Note that the uniform convergence assumption and f_n, g_n are bounded imply $\|f_n\| \leq M$, $\|g_n\| \leq N$ for all $n \geq 1$ for some M, N. For $\varepsilon > 0$, there is some n_1, n_2 such that $|f_n(x) - f(x)| < \varepsilon/(2 + 2N)$ for $n \geq n_1$ and $|g_n(x) - g(x)| < \varepsilon/(2 + 2M)$ for all $n \geq n_2$ and $x \in E$. It follows that

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq N|f_n(x) - f(x)| + M|g_n(x) - g(x)| \\ &< N\frac{\varepsilon}{2 + 2N} + M\frac{\varepsilon}{2 + 2M} \leq \varepsilon , \quad n \geq n_0 \equiv \max\{n_1, n_2\} . \end{aligned}$$

(b) We have

$$\|f_n^2(x) - f^2(x)\| = \left\| \frac{2x}{n} + \frac{1}{n^2} \right\| = \infty ,$$

so the convergence is pointwise but not uniform.

8. Let $f_n \rightrightarrows f$ on $[a, b]$ and $\|f_n\| \leq M$. For every continuous function Φ on $[-M, M]$, show that $\Phi \circ f_n \rightrightarrows \Phi \circ f$ on $[a, b]$.

Solution. By the uniform continuity of Φ over $[-M, M]$, for every $\varepsilon > 0$, there is some δ such that $|\Phi(z_1) - \Phi(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta$ in $[-M, M]$. For this δ, there associates an n_0 such that $|f_n(x) - f(x)| < \delta$ for all $x \in E$ and $n \geq n_0$. Thus,

$$|\Phi(f_n(x)) - \Phi(f(x))| < \varepsilon , \quad \forall n \geq n_0, \ x \in E .$$

That is, $\Phi \circ f_n \rightrightarrows \Phi \circ f$ on E.

9. Show that the following sequences are not uniformly convergent:

(a) $\{(\cos \pi x)^{2n}\}$ on $[a, b], a < b$, where $[a, b] \cap \mathbb{Z} \neq \phi$.

(b) $\{(x - 2)^{1/n}\}$ on $[2, 5]$

(c) $\left\{ \tan\left(\frac{n\pi x}{1 + 2n}\right) \right\}$ on $(0, 1)$.

Solution. (a) The pointwise limit is $f(x) = 0$ for non-integer x and $f(x) = 1$ for integer x. Therefore, once the interval contains an integer, f cannot be continuous. On the other hand, the sequence belongs to $C(\mathbb{R})$ and uniform convergence preserves continuity. Now, as the limit is discontinuous, the convergence cannot be uniform.

(b) This is just like the case $\{x^{1/n}\}$ we did in class. The pointwise limit is the discontinuous function $g(x) = 1, x \in (2, 5]$ and $g(2) = 0$.

(c) The pointwise limit is $f(x) = \tan(\pi x/2)$ which is unbounded on $(0, 1)$, but each function in this sequence belongs to $B((0, 1))$. Since uniform convergence preserves boundedness, the convergence cannot be uniform.

10. Let $f_n \in C[a, b]$ converge pointwisely to f on $[a, b]$. Suppose that $f_n \rightrightarrows f$ on (a, b). Show that $f \in C[a, b]$ and $f_n \rightrightarrows f$ on $[a, b]$.

Solution. Since uniform convergence preserves continuity, $f \in C(a,b)$. For $\varepsilon > 0$, there exists some n_0 such that

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3} , \quad \forall n \geq n_0,\ x \in (a,b) .$$

There is another n_1 such that $|f(a) - f_n(a)| < \varepsilon/3$ for all $n \geq n_1$. It follows that

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\varepsilon}{3} + |f_n(x) - f_n(a)| + \frac{\varepsilon}{3} , \quad \forall x \in (a,b),\ n \geq n_2 \equiv \max\{n_0, n_1\}. \end{aligned}$$

Taking $n = n_2$, since f_{n_2} is continuous at a, there is some $\delta > 0$ such that $|f_{n_2}(x) - f_{n_2}(a)| < \varepsilon/3$ for $|x - a| < \delta$. Therefore, for $x \in (a, a+\delta)$, we have

$$|f(x) - f(a)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon ,$$

hence f is continuous at a. Similarly, it is continuous at b.

11. Let $\{x_k\}$ be an enumeration of all rational numbers in $[0,1]$. Define $h_n(x)$ to be 1 at $x = x_1, \ldots, x_n$ and to be zero otherwise. Using this sequence to show that pointwise limit of integrable functions may not be integrable.

    Solution. The pointwise limit is the function $h(x) = 1$ if x is a rational number and $h(x) = 0$ if it is irrational. This function is not Riemann integrable.